VORONOI (THIESSEN) POLYGONS

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(continued inside back cover)

CONCEPTS AND TECHNIQUES IN MODERN GEOGRAPHY No. 45

VORONOI (THIESSEN) POLYGONS

by

B.N. BOOTS

(Wilfred Laurier University)

1. INTRODUCTION

Consider Figure 1a which represents a rectangular-shaped local authority area. The five open squares represent the locations of out-patient clinics, while the dots represent the locations of the homes of the out-patients. Suppose that you have been asked to assign each out-patient to one of the clinics. Each clinic is assumed to have no limit on the number of patients it can handle. Further, it is assumed that the cost of attending a clinic is borne by the patient and that such a cost, together with the time taken to travel to a clinic, is directly proportional to the distance between the patient's home and the clinic. In view of these assumptions you wish to provide a solution which allocates each patient to the nearest clinic. How can this be done? One way is to partition the local authority into five areas, each containing a clinic, so that for everywhere inside an individual area the distance to the clinic it contains is smaller than the distance to any other clinic. The problem of assigning a patient to a clinic then becomes simply one of examining in which of the five areas the patient is located. This solution is illustrated in Figure 1b. The areas demarcated in this figure are, in fact, the Thiessen polygons of the clinic locations.

While many geographers are familiar with the above and other basic uses of Thiessen polygons, fewer are aware of their links with other geographical concepts and techniques. In addition, many geographers are unaware of their widespread use (albeit under various aliases) in a diverse range of other disciplines including molecular physics, astrophysics, materials science, biochemistry, geology, ecology and archaeology. Much of the proliferation of Thiessen polygons stems from their application in new contexts and from recent extensions of the basic concepts. Their use now is such that the statistician Sibson (1980, p. 20) argues '... it seems reasonable to claim that [they are] one of the most basic and useful invariants associated with a point pattern in the plane.'

The purpose of this monograph is to examine the different aspects, both old and new, of Thiessen polygons and to explore their applications in geographical contexts. We begin this endeavour by formally defining Thiessen polygons.

2. DEFINITION

Consider a set, S, of n labelled points in the plane (see Figure 2a) where

$$S = \{p_1, p_2, \ldots, p_n\}$$

With each point, p_i, in S we associate all locations, x, in the plane which are closer to p_i than to any other point, p_j, in S ($j \neq i$). The result is to create a Thiessen polygon, P_i (see Figure 2b). More formally, if $d(x,i)$ is the euclidean distance from x to p_i, then

$$P_i = \{x \mid d(x,i) \leqslant d(x,j);\ j \in S,\ j \neq i\} \qquad (1)$$

It is possible that x is equidistant from a pair of points in which case it will lie on the boundary of P_i. In addition, x may be equidistant from three

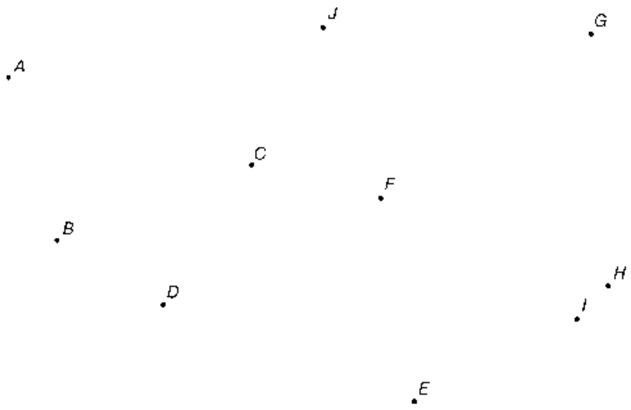


Figure 2a. A set, S, of labelled points in the plane.

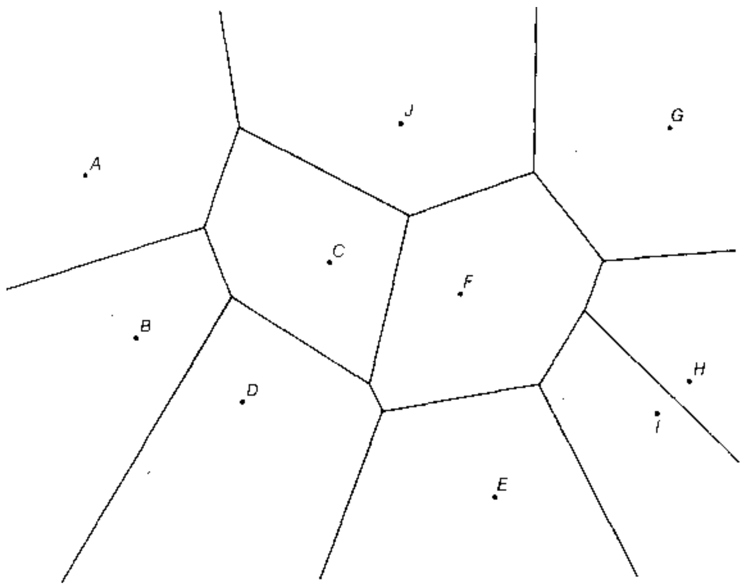


Figure 2b. The Voronoi diagram V(S) of S in Figure 2a.

The most obvious direct use is in developing spatial process models used in analyzing patterns of areas (see Section 6). They have also proved to be useful in the analysis of point patterns where several different approaches using Voronoi polygons have been developed (see Section 7). Finally, they have been used to provide solutions to some facilities location problems and examples of such uses are found in Sections 1, 6.1, 9.3 and 9.4. Examples of the use of Voronoi polygons in other disciplines than geography are given in Appendix B.

4. RANDOM VORONOI POLYGONS

Although Voronoi polygons occur in a wide range of disciplines and are used in a variety of applications, there is one type of Voronoi diagram which is common to most areas and consequently is quite frequently encountered. This is the diagram consisting of random Voronoi polygons (also known as the S-mosaic (Pielou, 1969, p. 145) or the random sets mosaic (Matern, 1960) in ecology.

4.1 Definition

Random Voronoi polygons are those associated with points located at random in the plane by a homogeneous planar Poisson point process (HPPPP). Despite the name this is a very simple process. It involves generating points in the plane subject to two conditions: (i) each location in the plane has an equal chance of receiving a point (uniformity assumption); and (ii) the selection of a location for a point in no way influences the selection of locations for any other points (independence assumption). Alternatively, these assumptions may be interpreted respectively as implying that the plane is completely homogeneous in all respects and that there is no interaction between the points. Hence, the pattern of points which results from the HPPPP can be considered as one which would occur by chance in a completely undifferentiated environment. With these assumptions, the probability, P(x), of x points occurring in a sample area of the plane is given by

$$P(x) = (e^{-\lambda} \lambda^{x}) / x! \qquad \text{for } x = 0, 1, \ldots \qquad (2)$$

where λ is the expected (average) number of points per sample area.

4.2 Properties

Despite the simplicity of the process which generates random Voronoi polygons, properties of such polygons have proved notoriously difficult to derive analytically and at present are limited to a few moment measures (see Table 1), although Solomon and Stephens (1980) have suggested a method by which distributions of such properties might be approximated. Consequently, researchers have relied heavily on Monte Carlo simulation procedures to derive additional results. The most extensive published results are those of Crain (1978) derived from studies of between 25 000 and 57 000 individual polygons, Quine and Watson (1984) from 50 000 polygons and Hinde and Miles (1980) using two million polygons.

In particular, Hinde and Miles (1980) obtain estimates of additional moments and the distribution of N, the number of sides (or vertices), s, a standardized measure of the length of the perimeter, S, (where $s = \lambda^{1/2} S/4$), a, a standardized measure of area, A, (where $a = \lambda A$) and θ_N, the N interior polygon angles. In addition, they estimate the covariances and correlation coefficients of pairs of N, s and a. Only the distribution of N is reproduced here (see Table 2).

Table 1. Moments for properties of random Voronoi polygons.

Property		Expected value
Number of sides (or vertices)	$E(N)$	6
Length of perimeter	$E(S)$	$4\lambda^{-1/2}$
Length of a side	$E(L)$	$2/3\lambda^{-1/2}$
Area	$E(A)$	λ^{-1}
	$E(A^2)$	$1.280\lambda^{-2}$

λ = mean number of points per unit area in the Poisson process.

(Sources: Evans, 1945; Meijering, 1953; Gilbert, 1962)

Table 2. Estimates of the values of N, the number of sides (or vertices) of random Voronoi polygons.

N	Estimated probability
3	0.011314
4	0.107123
5	0.259126
6	0.294406
7	0.199133
8	0.090161
9	0.029531
10	0.007429
11	0.001492
12	0.000246
13	0.000034
14	0.000005

(Source: Hinde and Miles, 1980, Table III, p. 215).

However, there remains one major limitation to the above results. They are aspatial. Although they describe the typical composition of a tessellation of random Voronoi polygons they reveal nothing about how the individual polygons are arranged within the tessellation. With this in mind Boots and Murdoch (1983) have extended Crain's and Hinde and Miles' results by examining aspects of the spatial arrangement of values of N, s and a. In particular, they estimate m_N, r_s and z_a, the average values of N, s and a, respectively, for those polygons adjoining an individual one of a given value of N, s and a.

4.3 Examples

Because of the assumptions involved in generating random Voronoi polygons, properties of them are often used as normative values in the analysis of empirical patterns. In geography, Boots (1973, 1975a) has used random Voronoi polygons to evaluate hypotheses relating to the formation of bus service centre hinterlands in two areas of the British Isles while Singh (1979) has used them in the analysis of village territories in the middle Ganga valley in India.

As an example of the normative use of random Voronoi polygons we will examine a pattern of civil parishes from central Dorset in south western England. The area considered is one of about 256 square miles on relatively homogeneous topography and its location is shown in Figure 3. The network itself is illustrated in Figure 4. The number of neighbours, N, was determined for each parish which is not truncated by the boundary of the sample area. The resulting values are summarized in column (2) of Table 3. Column (3) of this table also gives the expected values of N for the random Voronoi polygons calculated from the probabilities given in Table 2. The observed and expected values of N are compared using a chi-square goodness-of-fit test. This test reveals that the two sets of values are significantly different at the 95 per cent confidence level, primarily because there are more actual parishes with both low and high values of N. Thus, we conclude that, at least in terms of the numbers of neighbours, the parishes are significantly different from random Voronoi polygons. This, in turn, suggests that it is appropriate to search for a process for creating the parishes which is different to that involved in generating random Voronoi polygons. Further discussion of the use of Voronoi polygons in the analysis of patterns of areas occurs in Section 6.

5. DATA MANIPULATION

Here we discuss the different ways that Voronoi polygons have been used to facilitate the presentation and analysis of data. Probably, the earliest use in this regard was by Thiessen (1911) who wished to estimate the average rainfall of a region from a set of values recorded at individual stations in the region. Since the distribution of stations was not uniform over the region, he thought it inappropriate to simply use the average value calculated over the recording stations as the regional average. Instead, he suggested generating Voronoi polygons around each of the recording stations and then weighting each station's contribution to the regional average relative to the size of its associated polygon. Thus he used the following formula to calculate the regional precipitation, R,

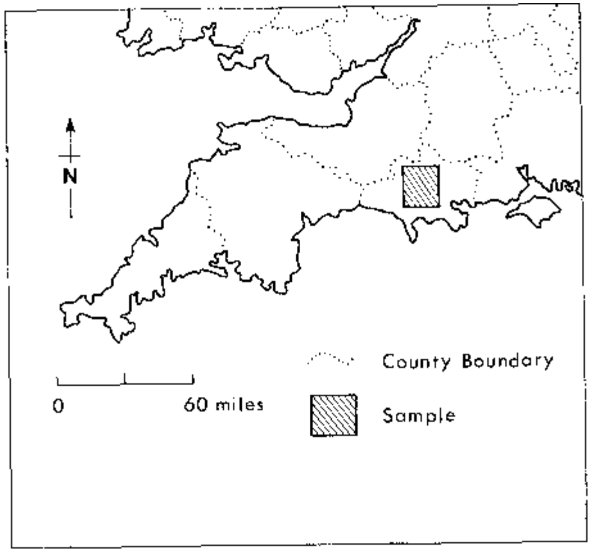


Figure 3.
Location of parish network.

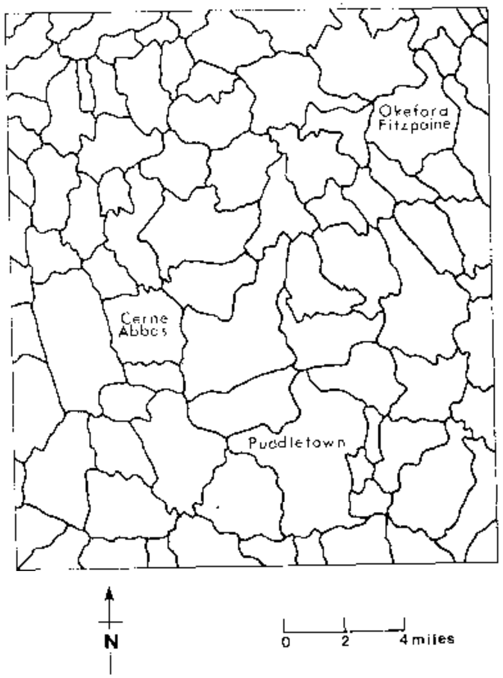


Figure 4.
Civil parishes in central Dorset (Compiled from *The Administrative County of Dorset;* Director General of the Ordnance Survey, Chessington, Surrey, revised edition, 1965.

Table 3. Number of sides, N, of civil parishes in central Dorset.

N (1)	Observed value (2)		Expected value (3)	
3	2	13	0.60	6.28
4	11		5.68	
5	8		13.73	
6	15		15.60	
7	6		10.55	
8	7	11	4.78	6.84
9	2		1.57	
10	1		0.39	
≥11	1		0.10	
Total	53		53	

$\chi^2 = 14.10$ df = 4 $\chi^2_{0.05} = 9.49$

$$R = (a_1 P_1 + a_2 P_2 + \ldots a_n P_n) / A \qquad (3)$$

where a is the area of the polygon associated with recording station i and $A = \sum_{i=1}^{n} a_i$.

Shortly afterwards Horton (1917) suggested a modification of this technique for obtaining rainfall estimates. His 'inclined plane' method involved defining triangles of points rather than polygons with respect to the recording stations although the specific procedure he described for identifying the triangles was somewhat arbitrary. Building on Horton's work, Whitney (1929) proposed using triangles which were equivalent to those defined by linking together recording stations whose Voronoi polygons share a common edge. We shall see below (see Section 7.1) that the network of triangles so formed is usually referred to as the Delaunay triangulation. However, there do not appear to have been any further published applications of Whitney's technique and it was not until about thirty years later, with the development of computer cartography, that the Delaunay triangulation began to be widely used in the display and manipulation of data.

Most of the early computerized procedures for drawing isarithmic maps from a set of random data points involved using the values at the data points to interpolate values on a regular grid covering the area to be mapped. The interpolated grid values were then used in drawing the isarithms. In such procedures the original data values often were not exactly represented in the grid or on the surface produced from it. An alternative approach is to represent the surface to be contoured by a triangulation of the data points.

The three vertices of each triangle define a plane so that an isarithm can be traced by using linear interpolation to identify the locations of the intersection of the isarithm with the edges of the triangulation and then linking these intersections (Peucker 1980; Dudycha, 1981, p. 126; Monmonier, 1982, pp. 154-156). The triangulation approach ensures that data points lie on the computed surface and is computationally faster than the interpolation-to-grid approach. Various ways of constructing the triangulation have been suggested (Gold *et al.* 1977; Yoeli, 1977; Elfick, 1979) but Peucker *et al.*, (1978) andMcCullaghand Ross (1980) argue that the Delaunay triangulation is most appropriate. This is because, besides being unique for a given set of data points, it simultaneously maximizes the number of triangles and produces triangles that are as equiangular as possible. This is important in the interpolation process since it ensures that triangle edge lengths and thus the distances between interpolation points are minimized. As an example of this procedure assume that each of the points in Figure 2a has an associated data value as shown in Figure 5a and it is our task to locate the 50 unit isarithm. We begin by creating the Delaunay triangulation (see Figure 5b). Then select any triangle in the triangulation through which the 50 unit isarithm must pass, say ABC. The isarithm must intersect two of the edges of this triangle. Select one of these edges, say AC and determine the point of intersection by linear interpolation using the values of the end points of AC, 44 and 51, respectively. Do the same for the other edge, BC, of ABC. Linking the intersection points on AC and BC provides the path of the isarithm through ABC. Then select one of the triangles adjacent to ABC, either ACJ or BCD and repeat the procedure. Continue until the path of the isarithm is traced. Since this procedure usually produces an unnaturally 'angular' isarithm, it is usually smoothed in some way.

Recently, Cromley (1984) has suggested that, since there is a strong correspondence between the structure of a choropleth map and V(S), the Delaunay triangulation of V(S) can be used to store choropleth base maps. Note that in both this procedure and that for drawing isarithmic maps it is necessary to know the spatial relations between the individual triangles in the triangulation. Such information is not necessary if the data points and their associated triangles are used in hill shading (Peucker *et al.* 1977; Brassel and Utano, 1979).

More fundamentally, Griffith (1982, p. 181) suggests that Voronoi polygons can be used to uncover the '... spatial infrastructure affiliated with a geographic distribution of points.' Consequently, he uses the Voronoi neighbour relationships as surrogates for contiguity measures required in some spatial autocorrelation procedures. Similarly, Haining *et a/.* (1984) use Voronoi neighbour properties to identify which weather stations should be used to estimate missing values for rainfall data from Kansas and Nebraska, USA.

Finally, several researchers have observed that Voronoi polygons often provide good approximations of more complex areal patterns. In a series of articles (Singh and Singh, 1975; Singh, 1976; Singh and Singh, 1977, 1978, Singh, 1979), Singh and his associates have used Voronoi polygons to approximate village boundaries in their analyses of various spatial characteristics of villages in different areas of the Ganga Valley in India.

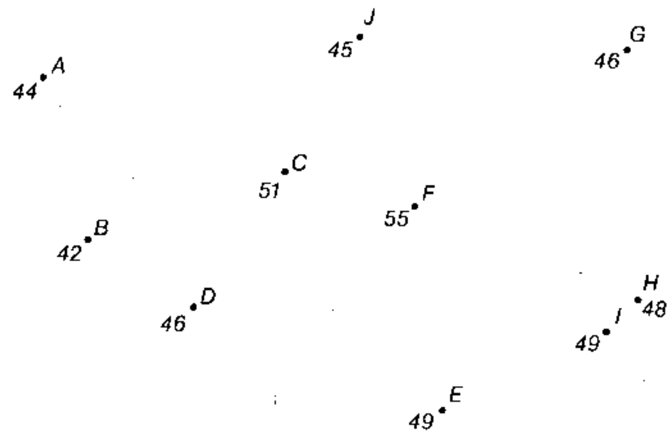


Figure 5a. Original data values.

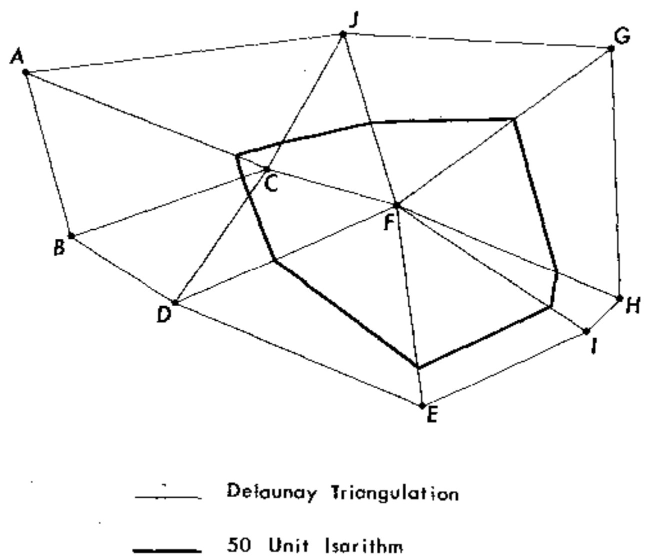


Figure 5b. Identification of 50 unit isarithm.

6. MODELS OF SPATIAL PROCESSES

Since the Voronoi diagram, V(S), produces a space-exhaustive partitioning of the plane into non-overlapping polygons, one of the most obvious direct applications of Voronoi polygons is in studying empirical patterns of areas which possess the same basic characteristics. In geography such area patterns are usually called cellular networks (Haggett, 1967). Most approaches involve equating the procedures involved in constructing V(S) with assumptions involved in simple spatial processes. In this way two main types of spatial models may be produced. These are assignment models and growth models.

6.1 Assignment Models

Assignment models describe processes which produce spatial patterns by allocating individual locations to particular points. In this case the construction of V(S) is equivalent to making the following assumptions:
(i) all points are present at the start of the assignment process; (ii) all points remain fixed in location throughout the assignment process; and (iii) any location in the plane is assigned to the nearest point. With these ideas in mind Cox and Agnew (1976) used Voronoi polygons defined around the existing county towns of Ireland to create a theoretical partition (see Figure 6a). They argue that such a partition represents an efficient allocation of locations to jurisdictions. The theoretical partition was then compared with the existing pattern of counties (see Figure 6b). Using an information theory measure they determine that the overall spatial correspondence between the two patterns is 72.46 per cent. However, the measure of agreement between corresponding individual counties in the two patterns is quite variable. Highest correspondences are for the larger coastal counties such as Clare (91.93 per cent) and Kerry (89.96) while lowest values occur for smaller inland counties such as Carlow (39.67 per cent) and Longford (43.79 per cent).

In a similar way Boots (1973, 1975a), after determining that the locations of bus service centres in two areas of the British Isles were essentially random, compared the properties of the centres' hinterlands with those of random Voronoi polygons (see Section 4).

As well as being used to analyze properties of known empirical patterns the assignment assumptions have been evoked in simulating patterns of areas where evidence is lacking on the actual empirical patterns. One of the earliest attempts to do this was by Bogue (1949) who used the Voronoi polygons defined around 67 US metropolitan centres as proxy for the real market areas.

As we illustrated in the Introduction, the assignment assumptions are also useful in location-allocation problems. To make the example of Section 1 more general, suppose we have a set of n facilities at predetermined sites in a region and that such facilities provide a service to a set of users also at predetermined locations in the same region (see Figure 1a). Obvious examples of such facilities are those providing municipal services such as health and education. In this situation one fundamental problem is how to allocate users to facilities. If our primary concern is to minimize the distances users travel to reach a facility (it is often assumed that

there is a strong correlation between distance travelled and other factors such as time and financial cost of the trip) and if we assume that there are no capacity constraints, the problem can be solved by defining Voronoi polygons around each facility and assigning users to the facility in whose Voronoi polygon they are located (see Figure 1b). This problem is usually referred to as the unconstrained transportation problem (Scott, 1971, p. 118; Keeney, 1972; Massam, 1972, pp. 9-10; 1975, pp. 59-60).

While such a formulation may be unrealistic because facility capacity constraints almost always occur in reality, it is often used as a norm by which to evaluate other proposed structures. Goodchild and Massam (1969) followed such a strategy in evaluating the efficiency of electrical service areas in southern Ontario, Canada.

Finally, we may note that the Voronoi polygon assignment assumption (iii) above is equivalent to the assumption in central place theory (Christaller, 1966; Losch, 1954) that consumers will satisfy their demands by visiting the nearest central place offering the required good or service. In both versions of central place theory the locations of service centres form a triangular grid over the plane. With this information in mind it may come as no surprise that when we generate the Voronoi polygons associated with these service centres they produce regular hexagons familiar as the market areas in central place theory. In other words the concept of market area in central place theory is identical to the concept of a Voronoi polygon. Dacey (1965) explores this relationship more fully and formally.

6.2 Growth Models

The procedures involved in constructing Voronoi polygons may also be equated with a set of simple growth assumptions. These are: (i) all points appear simultaneously; (ii) all points remain fixed in location throughout the growth process; (iii) for each polygon growth occurs at the same rate in all directions; (iv) the linear growth rate is the same for each polygon associated with a point; and (v) growth ceases for each polygon whenever and wherever it comes into contact with a neighbouring polygon.

Williams (1972, p. 152) has suggested that these assumptions could be used to model the development of depressions around stream sinks in karst topography while Smalley (1966) has used them to construct a model of crack patterns in basalt. Such patterns result from the contraction of lava flows on cooling. This contraction causes tensile stresses in the rock mass which in turn cause cracks. In particular, Smalley argues that tensile stress develops in a circular fashion about a set of stress centres on the cooling surface of basalt. In material of uniform temperature, stress centres appear simultaneously and form a regular arrangement so that growth resulting from them produces hexagonal cells. However, if uniformity of temperature is not present, as is usually the case, the centres will not develop simultaneously and will not be regularly arranged. Smalley suggests that instead they will be located at random (see Section 4.1) subject to the constraint that centres which would be closer than some specified distance to an already existing centre cannot develop. Stress centres are generated sequentially until no more can be located without violating the inter-centre distance constraint. Such a process is equivalent to sequentially locating a set of discs (of radius equal to half the inter-centre distance) at random in the plane such that the discs do not overlap (see Figure 7a) and is now

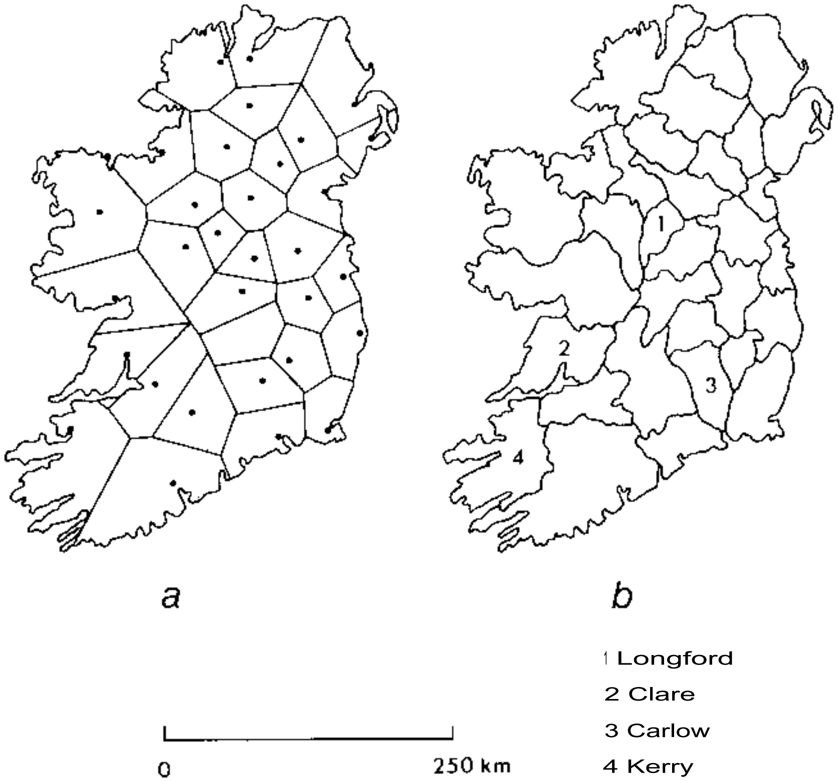
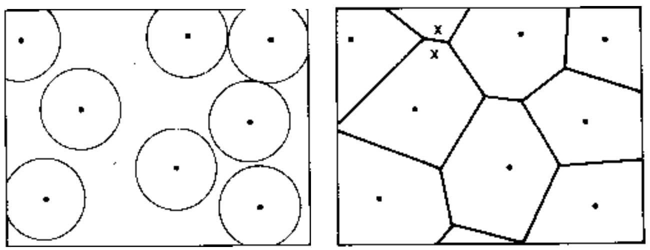


Figure 6a. Theoretical partition of Ireland. b. Actual partition.



a

Figure 7. Development of cracks in basalt a. location of stress centres.
b. final pattern.

usually referred to as a hard core process (Ripley, 1977). Once all the stress centres are located in this way, growth occurs according to assumptions (11)-(v) above. The realization of the process for the centres shown in Figure 7a is given in Figure 7b. Smalley ran a number of simulations of this model and examined the distribution of N, the number of neighbours of an individual cell. His values are given in column (3) of Table 4. These values were then compared with samples from four real world basalt flows. The values for one of these flows, Lewiston, Idaho, is given in column (2) of Table 4. A χ^2 test leads to the rejection (at the 95 per cent confidence level) of the Smalley model (see Table 4). Smalley suggests that this result might be due to the existence of very short edges in the model patterns (see Figure 7b). The counterparts of such edges in real-world lava flows may well have been overlooked in the compilations of field counts of N. Smalley thus suggests ignoring the short edges occurring in his model patterns and re-calculating the frequencies of N. The values for the modified model are given in column (4) of Table 4. These values offer a better fit to the data than the ones for the original model but a χ^2 test again leads to their rejection thus suggesting that the model may well be inappropriate for explaining the development of crack patterns in basalt.

Boots (1975a) has also used a growth model interpretation in his analysis of bus service centre hinterlands referred to in Section 6.1.

Table 4. Number of sides, N, of cells in basalt flows.

N (1)	Observed value (2)	Expected value: original model (3)		Expected value: modified model (4)
4	5	2.63	18.39	6.56
5	30	15.76		17.08
6	28	31.53		34.16
7	4	13.14	17.08	9.20
8	0	3.94		0
Total	67	67		67
χ^2		25.42		14.19
df		2		3
$\chi^2_{0.05}$		5.99		7.82

(Source: Calculated using data from Beard, 1959 and Smalley, 1966)

7. ANALYSIS OF POINT PATTERNS

We have seen that a Voronoi diagram, V(S), can be defined for any set of points, S, and that V(S) is unique. These observations have led a number of researchers (Evans, 1967; Haggett *et al.* 1977, pp. 436-439; Matern, 1979; Cormack, 1979, pp. 171-175; Ripley, 1981, p. 149; Cliff and Ord, 1981, p. 110; Thomas, 1981) to suggest that V(S) might be used in the analysis of point patterns.

In general, such analysis involves attempts to describe a point pattern objectively and to test hypotheses concerning possible generative processes for the pattern. In describing point patterns geographers most often use a three-fold typology. Such schemes involve establishing a benchmark pattern with respect to which other patterns are identified. The benchmark most frequently chosen is the pattern which results from the operation of the homogeneous planar Poisson point process (HPPPP) previously described in Section 4.1. As we noted there, a pattern resulting from such a process can be considered equivalent to one which would occur by chance in a completely undifferentiated environment. Such a pattern is usually called a random one.

The other two types of patterns are recognized relative to the random one. A clustered pattern is one in which the points are significantly more grouped in a region than they would be in a random pattern while in a regular pattern (sometimes also called a dispersed or uniform pattern) the points are more spread out over the environment than they would be in a random pattern. Just as we know that the HPPPP produces random patterns, it is possible to identify processes which give rise to clustered and regular patterns. (For examples of such processes see Haggett et al. 1977, Chapter 13, Getis and Boots, 1978; Ripley, 1981; Diggle, 1983; Upton and Fingleton, 1985).

Despite the various suggestions that V(S) might be used in point pattern analysis there appear to have been no attempts in geography to implement such suggestions. Instead, most attention has been focussed on a second tessellation which may be derived from V(S). In fact, this tessellation has already been introduced in Section 5 and we now turn our attention to it explicitly.

7.1 Delaunay Triangulation

Assume that we have created V(S) for a given set of points, S. If we join all pairs of points p_i, p_j $(i \neq j)$ in S whose Voronoi polygons, P_i, P_j, share a common edge we obtain a second tessellation. If three and only three edges are incident at each vertex in V(S), the new tessellation consists exclusively of triangles and is usually referred to as the Delaunay triangulation, D(S), of S (Delaunay, 1934; Rogers, 1964; Martin, 1965; Sibson, 1978). The incidence of more than three edges at a vertex in V(S) requires four or more neighbouring points in S to be on the circumference on the same circle whose interior contains none of the other points in S and so is extremely unlikely to occur in empirical patterns (see Figure 8 for an example). Such a situation is usually referred to as a degenerate case. The accompanying Delaunay 'triangulation' is also shown in Figure 8. The problem can be handled by either displacing one of the four points a small distance in any direction thus creating a short edge in place of the four edge vertex in

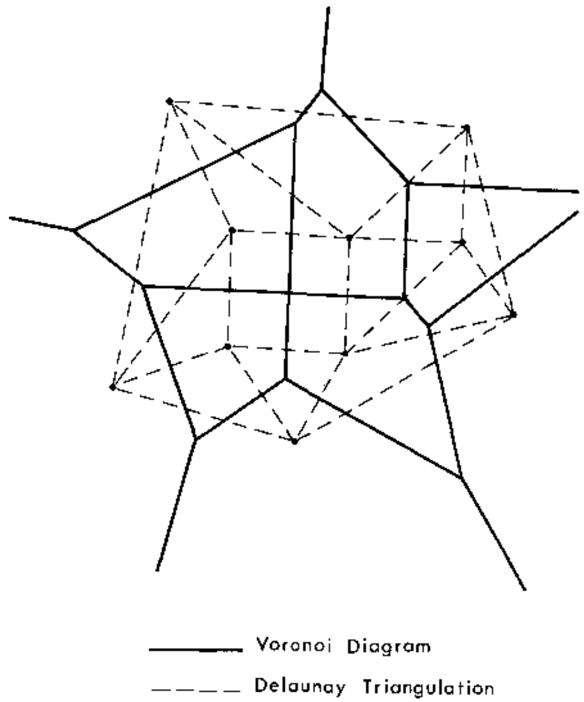


Figure 8. Portion of a Voronoi diagram containing a four-edge vertex and its associated Delaunay triangulation.

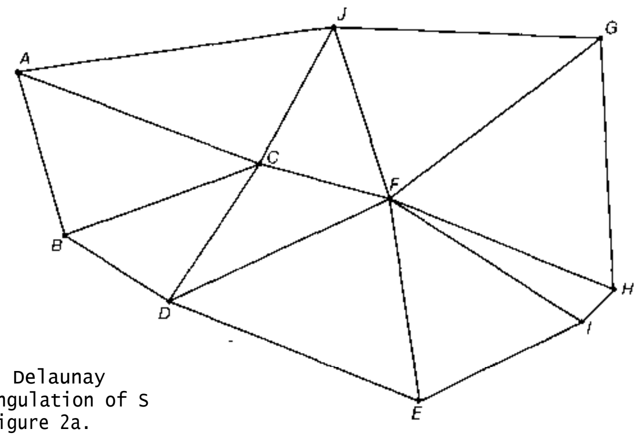


Figure 9. Delaunay Triangulation of S in Figure 2a.

V(S) (Vincent *et al.* 1983) or by drawing an arbitrarily selected diagonal of the four-sided figure in D(S) thus converting it into two triangles. The latter is probably preferable since it does not change the form of the individual polygons in V(S). D(S) for the set of points whose V(S) is shown in Figure 2b is illustrated in Figure 9. Note that unlike some of the polygons in V(S), none of the triangles in D(S) are unbounded and thus the outer edges of the triangulation define a convex figure which encloses S.

So far most attention has been focussed on the triangulation, D_P (S), associated with a random pattern of points (i.e., the realization of a HPPPP). Working independently Collins (1968, 1972) and Miles (1969, 1970) have obtained various properties of D_P (S) which are given in Table 5. They also derive the probability density function of a random angle, α, of a randomly selected triangle of D_P (S) as

$$f(\alpha) = [(\pi-\alpha)\cos\alpha + \sin\alpha]\,(4\sin\alpha/3);\quad 0 < \alpha < \pi \qquad (4)$$

The mean of α is, of course, $\pi/3$ and the second moment, $E(\alpha^2)$ is $[(2\pi^2)/9 - (5/6)]$.

Table 5. Moments for properties of $D_p(S)$

Property		Expected value
Perimeter	E(S)	$(32/3\pi)\lambda^{-1/2} = 3.395\lambda^{-1/2}$
	$E(S^2)$	$(125/3\pi)\lambda^{-1} = 13.263\lambda^{-1}$
Area	E(A)	$(1/2)\lambda^{-1} = 0.500\lambda^{-1}$
	$E(A^2)$	$(35/8\pi^2)\lambda^{-2} = 0.433\lambda^{-2}$

λ - expected number of points per unit area.

(Source: Miles, 1970)

Boots (1974, 1975b) and later Vincent *at al.* (1976, 1977, 1983) proposed using the above results to test if a given point pattern is significantly different from a random one. In particular, most attention has been paid to expression (4) since $f(\alpha)$ is independent of λ, the intensity of the HPPPP. The procedure is straightforward. First, we generate D(S) for the point pattern under investigation and then select a random sample of triangles from each of which we choose one angle at random. The values of these angles are then cumulated and this distribution is compared with the expected one derived from the indefinite integral of (4). Thus, the probability that a randomly selected angle of D_P (S) has a value less than or equal to x is

$$F(\alpha) = {}_0\!\int^x f(\alpha)d\alpha$$

$$= (1/3)\,\{2\sin^2\alpha + [\alpha\cos2\alpha - (3\sin2\alpha)/2 + 2\alpha]\,\pi^{-1}\}_0^x \qquad (5)$$

The goodness-of-fit between the empirical and expected distributions can be evaluated using an empirical distribution function (EDF) statistic such as the one-sample Kolmogorov-Smirnov (K-S) test. Both Boots (1975a) and Vincent *at al.* (1977) have used this procedure to analyze patterns of urban settlements in various parts of the USA which were originally examined by King (1961) using nearest neighbour analysis. However, Mardia *et al.* (1977) have

argued that this approach is wasteful of the available information and propose instead using the marginal density of the minimum angle, α_1, of the individual triangles which they derive as

$$f(\alpha_1) = (2/\pi)\,[(\pi-3\alpha_1)\sin2\alpha_1 + \cos2\alpha_1 - \cos4\alpha_1];\quad 0 \le \alpha_1 < \pi/3 \qquad (6)$$

A similar argument may also be made in favour of the use of the maximum angle, α_3, of the individual triangles. The marginal density of α_3 is given by the following expressions

$$f(\alpha_3) = (1/2\pi)\,(\sin4\alpha_3 + \sin2\alpha_3 + 2\pi\cos2\alpha_3 - 6\alpha_3\cos2\alpha_3)\qquad \pi/3 < \alpha_3 < \pi/2 \qquad (7)$$

and

$$f(\alpha_3) = (1/2\pi)\,(2\alpha_3\cos2\alpha_3 - 2\pi\cos2\alpha_3 - 3\sin2\alpha_3 + 4\alpha_3 - 2\pi)\qquad \pi/3 < \alpha_3 < \pi \qquad (8)$$

Preliminary work by Boots (1984) suggests that α_1 has the best all-round ability to reject a null hypothesis of a random pattern against both regular and clustered alternatives although there are indications that α_3 is able to detect some instances of clustering not discernable by α_1. In view of this result the procedure is illustrated for α_1 for a pattern of settlements in an area of south-eastern Montana, USA in 1973 (see Figure 10).

Using D(S) shown in Figure 10 we begin by identifying the minimum angle in each of the triangles (minimum angles for some of the triangles are illustrated in Figure 10). We can then obtain the proportion, $F'(\alpha_1)$, of minimum angles which do not exceed a given value of α. Note that in calculating $F'(\alpha_1)$, in order to avoid boundary effects, we ignore angles from those triangles where one or more of the vertices form one of the vertices of the outer boundary of D(S).[1] The values of $F'(\alpha_1)$ can then be compared with the corresponding expected values of $F(\alpha_1)$ for a random pattern using a one-sample K-S test. These expected values are derived from the indefinite integral of (6) and are given for values of a between 1 and 60 degrees (in one degree intervals) in Table 6. The procedure for this example is summarised in Table 7. In this instance the observed values are not significantly different from those expected from a random pattern. However, could we have proceeded further if the results had indicated a significant difference from the random values? The answer is a tentative yes. If the points were arranged in a perfectly regular way as in Figure 11 all the Voronoi polygons would be regular hexagons and all triangles in D(S) would be equilateral so that all angles would be 60 degrees. Thus, if for an empirical pattern there is an excess (relative to the proportions expected for a random pattern) of values at the upper tail of $f(\alpha_1)$, a regular pattern is indicated. Similarly, if the points are located so that they approximate a square grid (see Figure 12), the resulting polygons of V(S) will be all four-sided and the triangles of D(S) will be close to right-angled so that the minimum angles will be approximately 45 degrees. Thus, a significant excess of such angles indicates a pattern with similarities to a square grid. However, in a pattern with clusters of points some triangles will have edges which correspond to links between points on the peripheries of different clusters. Such triangles will

[1] Vincent *et al.* (1983) provide estimates of the magnitude of edge effects for samples of different sizes in a circular study area. They also suggest a strategy to avoid edge effects which could be used in some instances when the study area is of a regular shape.

be obtuse so that their minimum angles will be small. Thus, a distribution of α_1 with a significant excess of small angles is indicative of a pattern of clustered points.

Vincent and Howarth (1982) have extended further the analytical potential of this procedure by using Monte Carlo simulation techniques to estimate properties of D(S) associated with points located according to the hard core process described in Section 6.2.

7.2 Information Theory Approaches

Another approach to point pattern analysis involves using concepts developed in information theory. Information theory attempts to measure the degree of organization in a given system. If the system under investigation is a pattern of n points in a given region, E, it has been suggested that a measure of organization might be the variation in the frequency with which points

Table 6. Probability that the minimum angle of a triangle in $D_p(S)$ is less than or equal to a given value, α.

Radians α	Degrees	Probability $F(\alpha)$	Radians α	Degrees	Probability $F(\alpha)$
0.017453	1	0.000610	0.541043	31	0.500604
0.034906	2	0.002436	0.558496	32	0.526949
0.052359	3	0.005479	0.575949	33	0.553283
0.069812	4	0.009731	0.593402	34	0.579531
0.087265	5	0.015189	0.610855	35	0.605617
0.104718	6	0.021843	0.628308	36	0.631463
0.122171	7	0.029683	0.645761	37	0.656994
0.139624	8	0.038698	0.663214	38	0.682133
0.157077	9	0.048871	0.680667	39	0.706804
0.174530	10	0.060185	0.698120	40	0.730930
0.091983	11	0.072619	0.715573	41	0.754437
0.209436	12	0.086152	0.733026	42	0.777252
0.226889	13	0.100757	0.7504/9	43	0.799303
0.244342	14	0.116406	0.767932	44	0.820518
0.261795	15	0.133068	0.785385	45	0.840830
0.279248	16	0.150707	0.802838	46	0.860173
0.296701	17	0.169287	0.820291	47	0.878482
0.314154	18	0.188768	0.837744	48	0.895699
0.331607	19	0.209107	0.855197	49	0.911765
0.349060	20	0.230256	0.872650	50	0.926626
0.366513	21	0.252170	0.890103	51	0.940232
0.383966	22	0.274794	0.907556	52	0.952537
0.401419	23	0.298074	0.925009	53	0.963498
0.418872	24	0.321954	0.942462	54	0.973072
0.436325	'25	0.346374	0.959915	55	0.981240
0.453778	26	0.371271	0.977368	56	0.987960
0.471231	27	0.396582	0.994821	57	0.993212
0.488684	28	0.422240	1.012274	58	0.996977
0.506137	29	0.448177	1.029727	59	0.999243
0.523590	30	0.474322	1.047179	60	1.000000

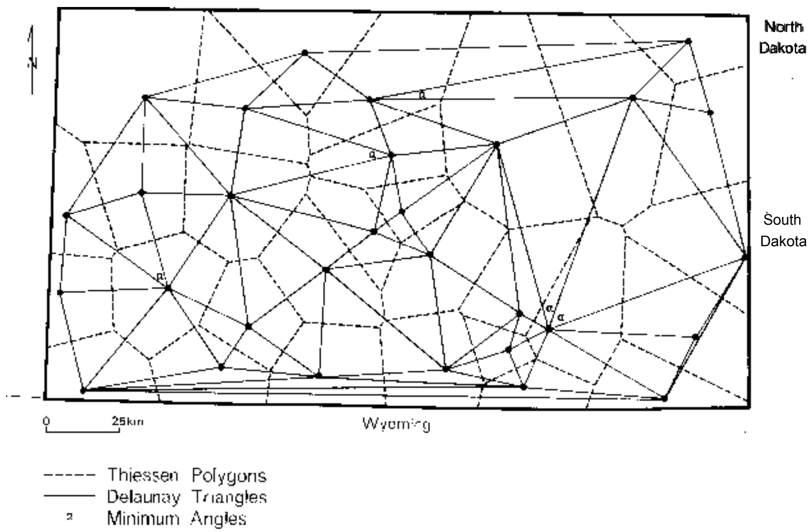


Figure 10. Voronoi diagram and Delaunay triangulation of a pattern of settlements in south-eastern Montana, USA.

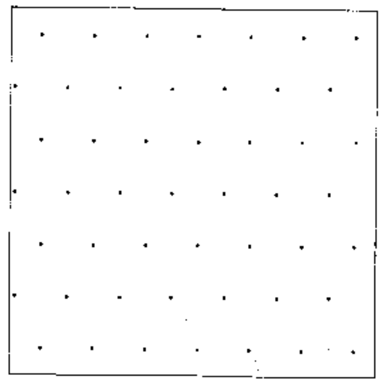


Figure 11. Portion of a triangular grid of points.

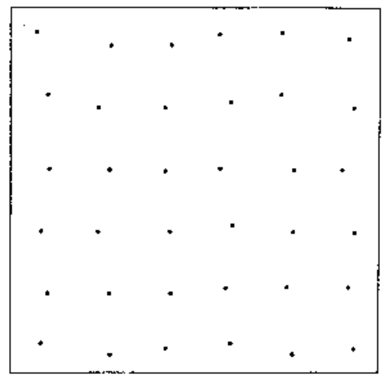


Figure 12. Portion of a pattern of points approximating a square grid.

Table 7. Analysis of locations of settlements in south-eastern Montana using the minimum angle technique.

α (in degrees)	Observed frequency	F'(α)	F (α)	|F'(α) - F(α)|
5	0	0.0000	.0152	.0152
10	2	0.0714	.0602	.0112
15	2	0.0714	.1331	.0617
20	4	0.1429	.2303	.0874
25	9	0.3214	.3464	.0250
30	13	0.4643	.4743	.0100
35	17	0.6071	.6056	.0015
40	20	0.7143	.7309	.0166
45	24	0.8571	.8408	.0163
50	27	0.9643	.9266	.0377
55	28	1.0000	.9812	.0188
60	28	1.0000	1.0000	-

D_{max} = 0.0874 Tabled D (α = 0.05) = 0.2570

occur in E. One measure of organization is the entropy statistic, H, given by

$$H = \sum_{i=1}^{n} x_i \ln (1/x_i) \quad (9)$$

where:

ln is the natural logarithm

x_i is described below

$\sum_i x_i = 1$

Chapman (1970) has suggested that (9) might be evaluated by defining the Voronoi polygon, P_i, for each point i and setting x_i equal to the ratio of the size, a_i, of P_i relative to the size, A, of E. However, in order to avoid edge effects we usually disregard any point for which one or more of the boundaries of its polygon intersects with the boundary of E so that x_i is usually calculated using

$$x_i = a_i \Big/ \sum_{i=1}^{n_c} a_i \quad (10)$$

where n_c is the number of points in E free from edge effects.

In fact, H, is more properly considered a measure of disorder since H_{max}, its maximum value occurs when we have no information other than n and A. In this case our best guess for any point in E would be that a_i is A/n. This would mean that x_i would equal 1/n for each of the n points so that

$$H_{max} = n \{1/n \ln [1/(1/n)]\} = \ln n \quad (11)$$

However, there are problems in using H directly to evaluate a given point pattern (Chapman, 1970, pp. 320-321; Lenz, 1979, pp. 376-377) and it is suggested that Theil's redundancy measure, R*, given by

$$R^* = \sum_i x_i \ln [x_i/(1/n)] = H_{max} - H \quad (12)$$

is more useful and easier to interpret. This is because R* = 0 for a regular pattern and increases as clustering of the points in E increases (Lenz, 1979, p. 377).

Thus, R* can be computed for an empirical pattern and compared to the expected value, E(R*), for a random pattern. Estimates of E(R*) have been obtained by Lenz (1979) and are given in Table 8. In addition, Lenz suggests that, if n ⩾ 15, the difference between R* and E(R*) can be tested using a normally distributed statistic, z, of the form

$$z = [R^* - E(R^*)] \Big/ \sqrt{var(R^*)} \quad (13)$$

Estimates of $\sqrt{var(R^*)}$ have also been derived by Lenz and are given in Table 8. Note that besides use in testing whether or not an empirical pattern is significantly different from a random one, since R* is independent of n, it can also be used to compare two empirical patterns. Thus, for example, if we had patterns of grocery stores in a given city at two different points in time we might use R* to determine whether the pattern had changed significantly over time and if so whether the clustering of stores had increased or decreased. Another example is provided by Lenz (1979) who compares the pattern of personal robbery location's in the city of Milwaukee in 1975 with the pattern of victim addresses. He finds that the former pattern shows more clustering than the latter and thus concludes that victims contribute to the occurrence of personal robberies by exposing themselves to high crime-risk areas of the city.

Table 8. Estimates of the expected values and standard deviations of Thiel's redundancy measure for a random pattern

N	E (R*)	√var(R*)
15	0.118	0.0389
30	0.122	0.0286
60	0.124	0.0211
90	0.125	0.0165
180	0.125	0.0123
360	0.125	0.0087
∞	0.126	0

(Source: Lenz, 1979)

8. OTHER RELATED STRUCTURES

In the previous section and in Section 5 we saw that it was possible to generate a second structure, the Delaunay triangulation, D(S), from the Voronoi diagram, V(S), of a set of points, S and that D(S) had a number of useful applications in data manipulation and point pattern analysis. Here we describe several other structures which are contained within D(S) and are thus related in some way to V(S). Although, as yet, these related structures have received only scant attention in geography, we shall see that they have been proposed and used by other disciplines in applications of interest to geographers. In general, they are concerned with responding to the problem that, given a set, S, of points in the plane, how can we create some structure for S by joining pairs of points in S? This question is of interest in such areas as cluster analysis including regionalization, spatial autocorrelation and perception.

8.1 Convex Hull

This is simply the smallest convex polygon which encloses the points in S. Figure 13 shows the convex hull, CH(S), of that set of points whose V(S) and D(S) are shown in Figures 2b and 9, respectively. Comparison of Figures 9 and 13 will indicate that CH(S) is defined by the outer edges of D(S). Further, it may be recalled that these edges link the boundary polygons of V(S), i.e., the unbounded polygons, shown in Figure 2b. Most frequently, CH(S) has been proposed as a crude approximation of the shape of the pattern of points in S.

8.2 Minimal Spanning Tree

If S consists of n points a set of n-1 links can be defined which join the n points so that it is possible to trace a path from any point to any other point. The set of links for which the aggregate length is a minimum defines the minimal spanning tree, MST(S), of S. The MST of the set of points whose D(S) is shown in Figure 9 is illustrated in Figure 14. Comparison of Figures 14 and 9 will reveal that all the links in MST(S) are contained in D(S).

The MST(S) can be constructed by first linking each point of S to its nearest neighbour (see Figure 15a). After doing this we are usually left with a set of disconnected elements, in this case six. We now link each disconnected element to its nearest neighbouring element (see Figure 15b). This reduces the number of disconnected elements. The procedure is then repeated until only one element, linking all the original points, remains (see Figure 15c). This is the MST(S).

The MST(S) is of importance in planning simple communications and transportation networks. It can be used as a normative model where the aim is to minimize the cost (which is usually assumed to be directly proportional to the length) of such a network subject only to the constraints that links may only intersect at points in S and that it must always be possible to trace a path between every pair of points using only the n-1 links.

8.3 Relative Neighbourhood Graph

In this structure devised by Toussaint (1980), following a suggestion by Lankford (1969), two points, p_i, p_j, in S are linked if they are at least as close to each other as they are to any other point in S, that is they are linked if

$$d(i,j) \leq \max[d(i,k),\ d(j,k)]\ \forall k;\ k \neq i,j \qquad (14)$$

Application of expression (14) to the set of points in Figure 2a reveals that, in this instance, the relative neighbourhood graph, RNG(S) is the same as the minimal spanning tree, MST(S) illustrated in Figure 14. However, as Figure 16 shows the minimal spanning tree and the relative neighbourhood graph of a set of points need not be the same. Note that both Figures 14 and 16 satisfy the conditions that every link in RNG(S) occurs in D(S) and that

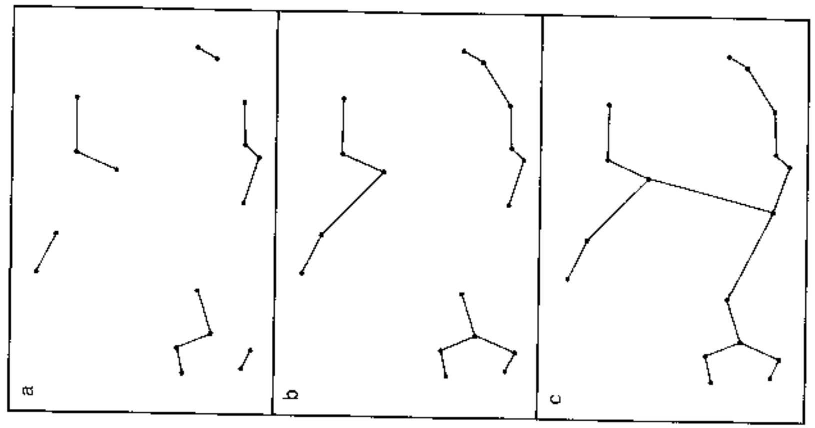


Figure 15. Construction of a minimal spanning tree.

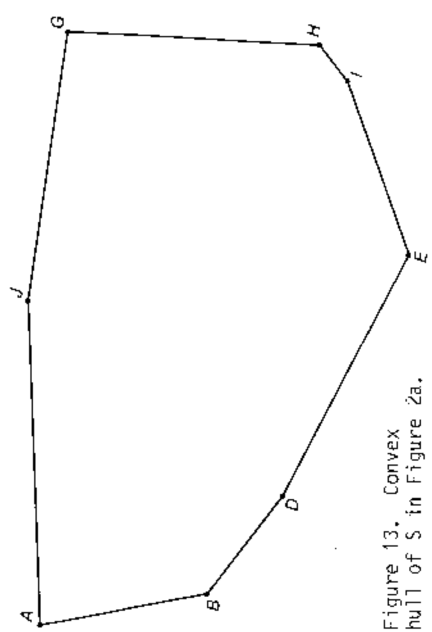


Figure 13. Convex hull of S in Figure 2a.

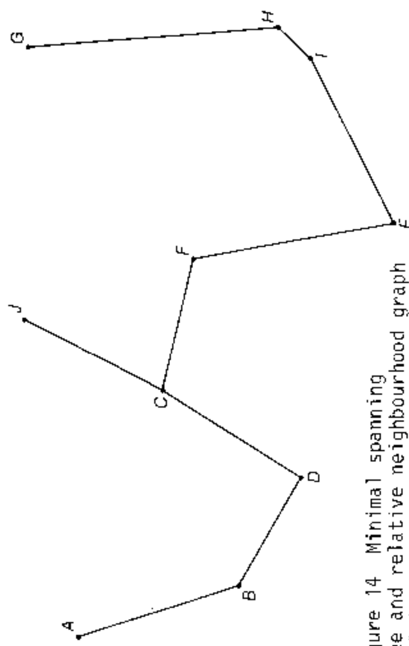


Figure 14 Minimal spanning tree and relative neighbourhood graph of S in Figure 2a.

all links in MST(S) appear in RNG(S).

8.4 Gabriel Graph

For any set of points, S, the Gabriel Graph, GG(S), can be formed from V(S). If the Voronoi polygons, P_i, P_j, of two points, i,j, are contiguous, i,j are linked in the Gabriel Graph if a line joining i and j intersects the common boundary segment of P_i, P_j at a point other than the end points of that segment (Matula and Sokal, 1980). Meijering (1953) calls the points which are linked in this way to any given point i, the full neighbours of i and demonstrates that on average the number of such points is four. Equivalently the links in a Gabriel Graph can be obtained by joining two points, p_i, p_j, distance, d(i,j) apart if

$$d^2(i,j) < [d^2(i,k) + d^2(j,k)] \forall k;\ k \neq i,j \qquad (15)$$

This, in turn, is equivalent to linking i,j if a circle passing through i,j contains no other points of S.

The GG(S) for the pattern whose V(S) and D(S) are given in Figures 2b and 9, respectively is shown in Figure 17. Comparison of Figure 17 with Figure 9 will indicate that all the links contained in GG(S) occur in D(S). In addition, comparison of Figures 17 and 14 shows that all the edges in RNG(S) (and thus MST(S)) occur in GG(S). The sequence of structures MST(S), RNG(S), GG(S) and D(S) thus represents one of increasing complexity in terms of the number of links between points in S and consequently provides alternative definitions of geographic connectivity. Despite frequent use in biology (see Appendix B) GG(S) does not appear to have been used by geographers.

9. EXTENSIONS

There are a number of ways in which the basic concepts involved in defining Voronoi polygons can be extended. Although Voronoi polygons have been known for almost a century and a half, most of the extensions have only been proposed within the last ten years and consequently there are relatively few applications of them. In the following sections we explore some of these extensions which appear to have potential geographical applications.

9.1 Weighted Voronoi Polygons

Here we maintain the basic assumption of a set of points, S, in the plane but in addition we assign a weight, w_i ($w_i > 0$), to each point, p_i, which relates to some variable property of the phenomenon represented by p_i (e.g., the population size of a settlement, the number of functions in a shopping centre, the amount of emissions from a polluter). We use these weights to define a weighted distance, $d_w(x,i)$, between any location, x, in the plane and p_i. The weighted polygon, WP_i, associated with p_i is then defined as follows,

$$WP_i = \{x \mid d_w(x,i) \leq d_w(x,j);\ j \subset S,\ i \neq j\} \qquad (16)$$

If WP_i is defined for all i, we produce a tessellation called the Weighted Voronoi Diagram, WV(S), of S.

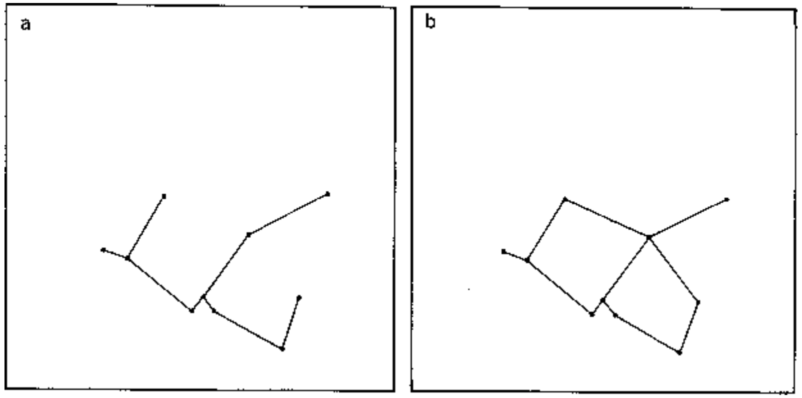


Figure 16a. Minimal spanning tree and b. relative neighbourhood graph of S in Figure 2a.

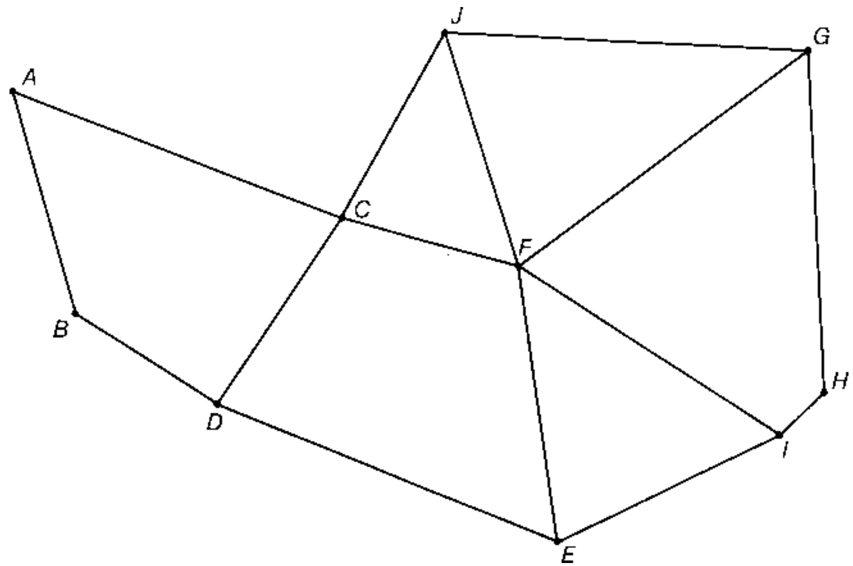


Figure 17. Gabriel graph of S in Figure 2a.

Various weighting schemes are possible, but first note that if all w_i are the same, expression (16) is the same as (1) in Section 2 and $WP_i = P_i$ and $WV(S) = V(S)$, and we revert to the basic Voronoi polygon model. If we define $d_w(x,i)$ as in expression (17),

$$d_w(x,i) = d(x,i) \,/\, w_i \tag{17}$$

the polygons in WV(S) have the following characteristics:

(i) they have either straight line segments (if $w_i = w_j$, $i \neq j$) or circular arcs ($w_i \neq w_j$, $i \neq j$) as edges; (ii) they are not necessarily convex; (iii) they may be entirely contained within another polygon; and (iv) they may be fragmented (i.e., may consist of two or more parts). WV(S) for the set S used to create V(S) in Figure 2b is shown in Figure 18. Although Figure 18 illustrates the occurrence of characteristics (3) and (4), they are more clearly shown in Figure 19.

The reader may note the similarity between the form of (17) and the family of gravity models which use an inverse power deterence function. In fact Boots (1980) demonstrates that the polygon edges created by using the weighting scheme in (17) are equivalent to those identified by the breakpoints and lines of equilibrium of Reilly and Huff-type models with an exponent of 1, respectively.

As with V(S), the construction process of WV(S) can be equated with spatial process models of either an assignment or growth type. In the case of the former the assignment process is one in which locations in the plane are assigned to points in proportion to the respective weights of those points. In the growth version of the model it is assumed that the linear rate of growth of the area around each point is the same in all directions and is directly proportional to the weight associated with the point.

The majority of applications of this weighting scheme is found in the work of Huff and his associates. They begin with theoretical treatments (Gambini *et al.* 1967; Huff and Jenks, 1968) and then apply their findings to two national urban systems. In Huff (1973) the model is used to demarcate urban spheres of influence for two levels of the USA urban system. The weights attached to the points (urban centres) are their factor scores on a dimension of functional size derived from a factor analysis of a 1762 x 97 data matrix (Berry, 1972).

Huff and Lutz (1979) also apply the model to the Republic of Ireland, suggesting that the resulting areas can be '... used to provide a geographical delivery system for serving public interest' (p. 196). They note that, while the existing county units of the Republic are used to provide many of the local government functions, many of them date back to the sixteenth and seventeenth centuries and may have become outdated because of subsequent cultural, social and economic changes. They begin their analysis by selecting the 114 urban centres which had populations of at least 1500 in 1970. Information was then collected for each of these centres for 33 socioeconomic variables. This data was factor analyzed yielding nine factors accounting for 85 per cent of the total variance. The most important factor (36 per cent of the total variance) was a measure of functional size. The factor scores from this dimension for the 114 centres were clustered into groups generating a five level urban hierarchy. The factor scores were also used as the weights in the model to define the spheres of influence of the centres

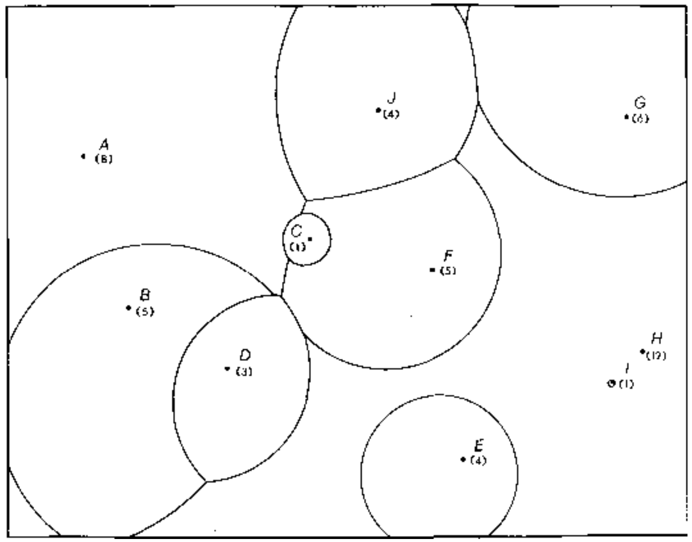


Figure 18. Weighted Voronoi diagram of S in Figure 2a. using definition of weighted distance in expression (17). Numbers in parentheses represent weights associated with points.

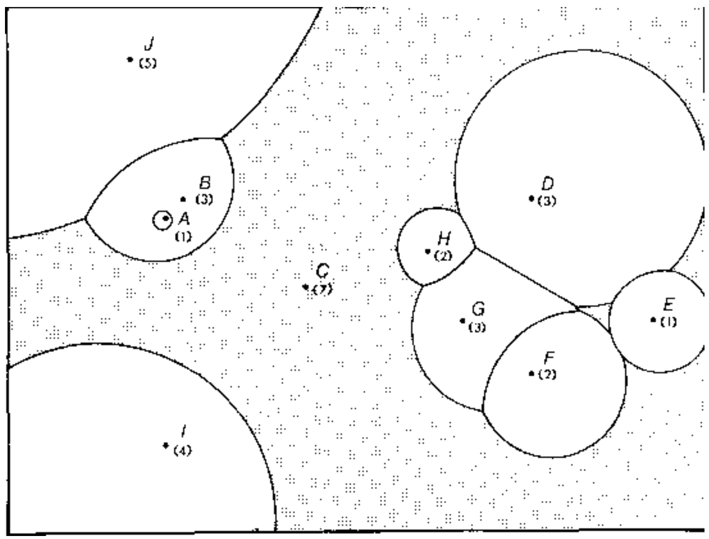


Figure 19. Weighted Voronoi diagram using definition of weighted distance in expression (17) and showing embedded and fragmented polygons. Numbers in parentheses represent weights associated with points. Shaded area identifies polygon associated with point, C.

at the different levels. The partition created for the 14 third-order centres is shown in Figure 20a. Since Huff and Lutz think that it is unlikely that the existing county structure will be abolished, the spheres of influence created by the model are redrawn so that they correspond with existing county boundaries (see Figure 20b). In this transformation a county is assigned to the urban centre whose sphere of influence contains the largest proportion of the county's area. However, if a sphere of influence of an urban centre encompasses less than the major portion of a county, it is eliminated, as is the smaller of two urban places in the same county. Thus, Dun Laoghaire, present in Figure 20a is absent from Figure 20b. Huff and Lutz suggest that patterns such as that in Figure 20b can be used to identify centres that are likely to have the greatest impact on depressed areas if investment is channeled through them. Thus, in the case of the Republic, where the undeveloped areas are mainly in the north-west and west of the country, they suggest that Galway and Sligo would be appropriate choices as growth poles.

Illeris (1967) and Hubbard (1970) also use the model to define functional regions for urban centres in Denmark and Jamaica, respectively, while Cox and Agnew (1974) use it to partition all of Ireland into theoretical counties which are then examined in terms of their relations to actual counties. They use the locations of the areal centroids of existing counties for point locations which are assigned weights in terms of their actual populations. Unfortunately, the map they produce is clearly in error not only because, as Weaire and Rivier (1984, p. 68) point out, they lose one of the thirty-two counties of Ireland.

A second possible weighting scheme is

$$d_w(x,i) = d(x,i) - w_i \tag{18}$$

In this case the polygons in WV(S) have the following characteristics

(i) either straight line segment ($w_i = w_j$, $i \neq j$) or hyperbolic arcs ($w_i \neq w_j$, $i \neq j$) as edges
(ii) are not necessarily convex
(iii) the common boundary between two polygons may be discontinuous.

WV(S) defined using the weighting scheme of (18) for the same set of points in Figure 2b and 18 is illustrated in Figure 21. If this procedure is equated with spatial process models, the assignment version is equivalent to one in which a circle of radius directly proportional to w_i is defined around each point and locations in the plane are assigned to the nearest circle. In the growth version of this model, initially a circle of radius directly proportional to w_i is defined about each point and these circles then grow in all directions at the same linear rate which is constant for all circles.

Such a weighting scheme ensures a minimum size for any WP_i and is consistent with the notion of a minimum threshold. It is interesting to note that Von Hohenbalken and West (1984a) have used a simplified version of this model to study predation amongst supermarkets in the city of Edmonton, Canada. Boots (1980) shows that the forms of polygons produced by this weighting scheme are identical to those of polygons produced by a dynamic spatial growth model usually referred to as the Johnson-Mehl model (Getis and Boots, 1978, Chapter 7).

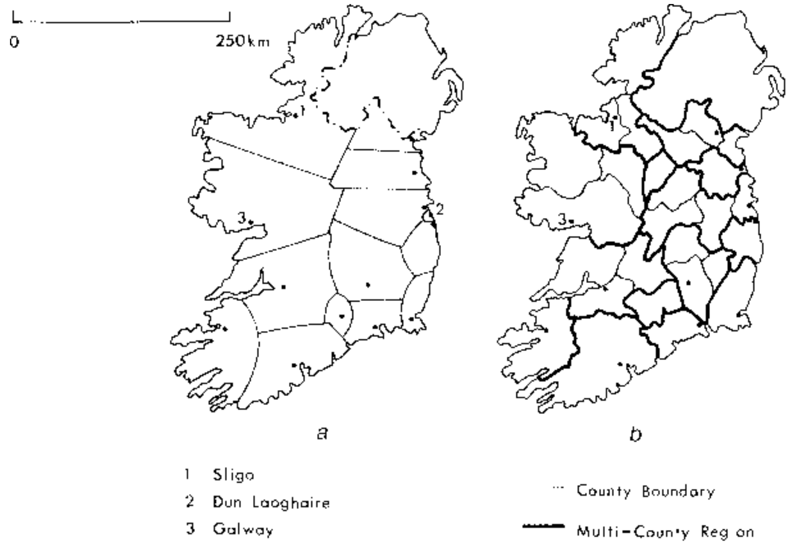


Figure 20. Republic of Ireland. a. Spheres of influence of third-order centres. b. Multicounty regions of third-order centres. (Redrawn from Huff and Lutz, 1979).

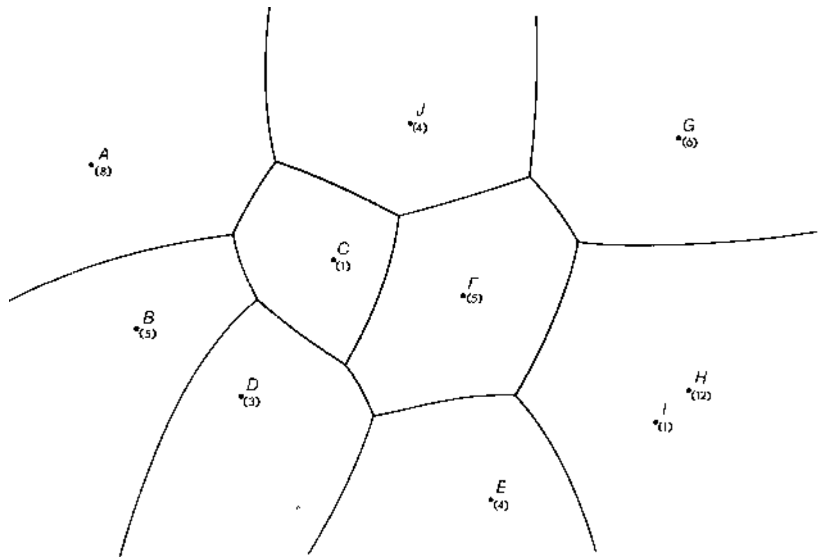


Figure 21. Weighted Voronoi diagram of S in Figure 2a using definition of weighted distance in expression (18). Numbers in parentheses represent weights associated with points.

Another possible weighting scheme is

$$d_w(x,i) = d^2(x,i) - w_i$$

This produces a WV(S) in which all polygon boundaries are straight line segments. In fact a common boundary between WP_i, WP_j is the locus of points from which the tangents to two circles, radii $\sqrt{w_i}$ and $\sqrt{w_j}$ centres, p_i, p_j, respectively, have equal length (Coxeter, 1961, pp. 85-87). This locus is known as the radical axis of the two circles. Alternatively, such loci are referred to as power lines since they can be defined using the concept of the power of a location with respect to a circle (Coxeter, 1961, p. 81; Aurenhammer, 1983).

The WV(S) formed by using expression (19) for the same point set, S, used in Figures 18 and 21 is shown in Figure 22. The individual polygons so formed are convex but it is possible that a given point, p_i, does not have an associated weighted polygon, WP_i, or that p_i is outside of WP_i. Figure 23 illustrates more clearly both of these possibilities.

When used as a spatial process model this weighting procedure is interesting in that once established, a circle associated with a point has a linear rate of growth which is inversely proportional to its diameter.[1] Such a situation would be consistent with a situation of decreasing returns to scale since the growth rate of individual polygons will decline with time. So far there have been no applications of this weighting scheme in geography, although it has been applied elsewhere (see Appendix B).

Finally, it should be noted various other definitions of $d_w(x,i)$ are possible many of which have been explored theoretically by Hanjoul et al. (1984).

9.2 Generalized Voronoi Polygons

So far we have considered Voronoi polygons associated with a set of points in the plane. This extension involves considering Voronoi polygons generated about other geometric features such as lines and circles. More formally, if $S = \{s_1, s_2, \ldots, s_n\}$ is a set of n labelled objects in the plane, and d(x,i) is the least Euclidean distance from any location, x, in the plane to any point of s_i, the generalized Voronoi polygon of s_i, GP_i, is defined as

$$GP_i = \{x \mid d(x,i) \leqslant d(x,j);\ j \in S,\ j \neq i\} \qquad (20)$$

The set of all such GP_i forms the generalized Voronoi diagram, GV(S), of S.

So far there has been little work in this area (see Appendix B). However, note that GV(S) for a set of circles is the same as that produced by the weighted polygon model of expression (18).

9.3 Order-k Voronoi Polygons

In this extension, we return to the situation of a set of points, S, in the plane, but instead of considering individual points of S we consider a subset, S_k, of S containing k labelled points. We then create a region $P(S_k)$ such that all locations in $P(S_k)$ are at least as close to all points in S_k than

[1] I am grateful to Raimund Seidel for pointing this out to me.

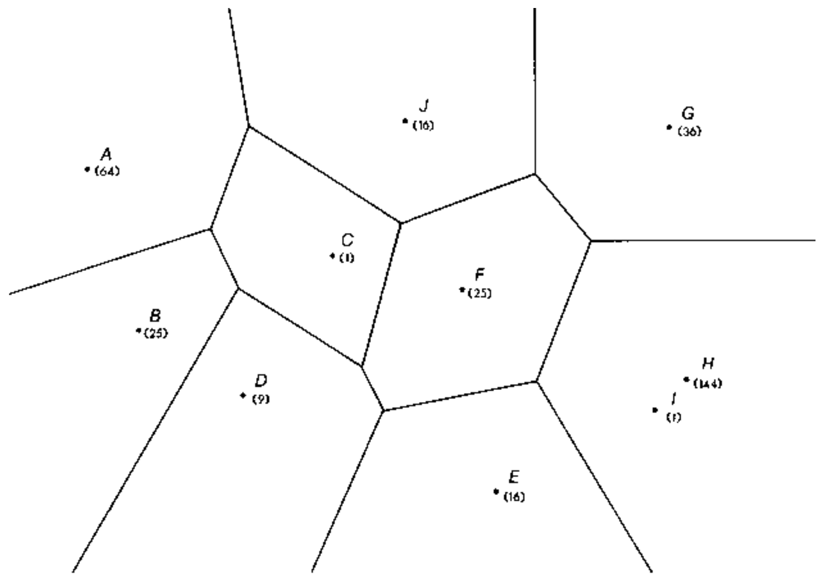


Figure 22. Weighted Voronoi diagram of S in Figure 2a using definition of weighted distance in expression (19). Numbers in parentheses represent weights associated with points.

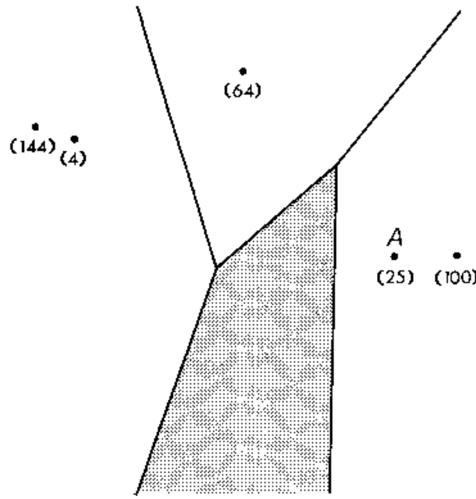


Figure 23. Special properties of weighted Voronoi diagram using definition of weighted distance in expression (19). Numbers in parentheses represent weights associated with points. Shaded area is polygon associated with point A.

they are to any point not in S_k. Thus, if x is a location in the plane and d(x,i) is the Euclidean distance from x to a point in S,

$$P(S_k) = \{x \mid \forall i \in S_k, \forall j \in S-S_k, d(x,i) \leq d(x,j)\} \qquad (21)$$

Obviously, when k=1 we have the basic Voronoi polygon model. However, when k>1 it is possible that $P(S_k)$ is empty, i.e., not all subsets of k points yield regions. Whether or not a particular subset of points generates a region depends on the relative location of points in S. However, the total number of such regions generated is of the order of k(n-k). As with the basic Voronoi polygons, all such polygons, $P(S_k)$, are convex and partition the plane exhaustively. This partition is called the Voronoi diagram of order -k, $V_k(S)$, of S. $V_k(S)$ for k=2 for the set of points in Figure 2a is illustrated in Figure 24. Note that $P(S_k)$ need not contain k points and may contain only one or no points of S. Miles (1970) (see also Miles 1972 and Miles and Maillardet, 1982) has obtained the mean values of the number of sides (vertices), perimeter and area of individual polygons in $V_k(S)$ (k = 2, 3, ...) for points located according to a homogeneous planar Poisson point process (see Section 4.1).

How can $V_k(S)$ be constructed? Suppose we already have the basic Voronoi diagram, V(S) of S and we consider a test location, q, in the plane (see Figure 25). We can immediately determine which point of S is closest to q by simply observing in which polygon of V(S) it is located. It follows that the second closest point of S to q must be one of those points whose Voronoi polygons are adjacent to that in which q is located. Thus, if we could partition this host Voronoi polygon into subregions each of which is adjacent to one of the surrounding polygons, the location of q in these subregions would identify the second closest member of S. If we carry out this partitioning process for all polygons in V(S) the new edges so produced yield V_2 (S) shown in Figure 24. Similarly, we can partition the polygons in $V_2(S)$ to produce $V_3(S)$ and so on. We can obtain the partition for a particular polygon of V(S) by removing from S the point associated with the polygon in question and generating the V(S) for the remaining points (see Figure 26). Thus, V_2 (S) can be created from V(S) by successively deleting each point of S and solving V(S) for the remainder. The union of the new edges so produced yield V_2 (S).

Lee (1982) demonstrates that each new vertex created in an order-k Voronoi diagram corresponds to a circumcentre of three points of S and that the circumcircle so obtained contains exactly k-1 points of S in its interic Thus, for the basic Voronoi polygon model where k=1, the circumcircle will be empty. Shamos and Hoey (1975) show that this fact is useful in solving a particular facilities location problem. This is one in which the goal is to add a new facility to an existing set of facilities such that the new facility is as far away as possible from any existing facility. Such a strategy might be adopted if the facility was a shopping mall and we wished to minimize spatial competition with existing malls or if the facilities produced pollutants and we wished to diffuse the overall pollution as much as possible. The •problem may be solved by identifying that vertex of V(S) with the largest circumcircle (see Figure 27).

In general, knowledge of V_k (S) is useful in those location problems where we wish to identify the second, third, etc. closest facility to a particular location. One such instance is in the reassignment of serviced locations when a facility is eliminated. Another occurs in the case of

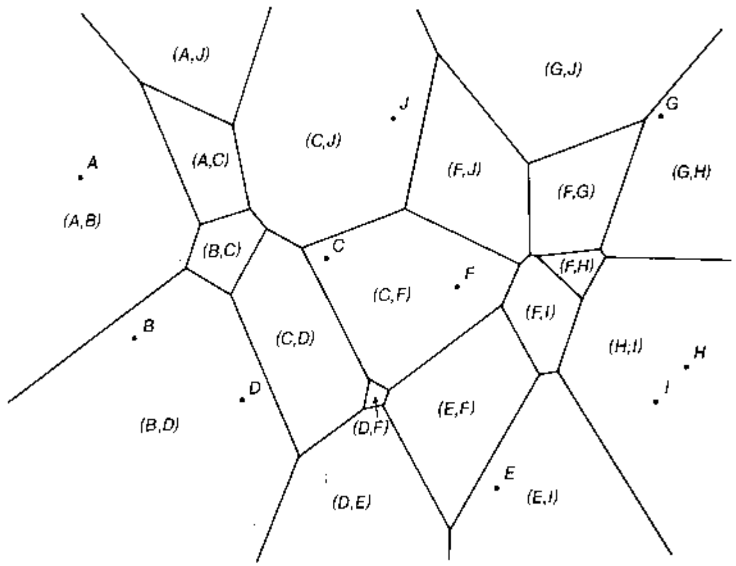


Figure 24. Order-2 Voronoi diagram of S in Figure 2a. Letters in parentheses indicate points associated with each polygon.

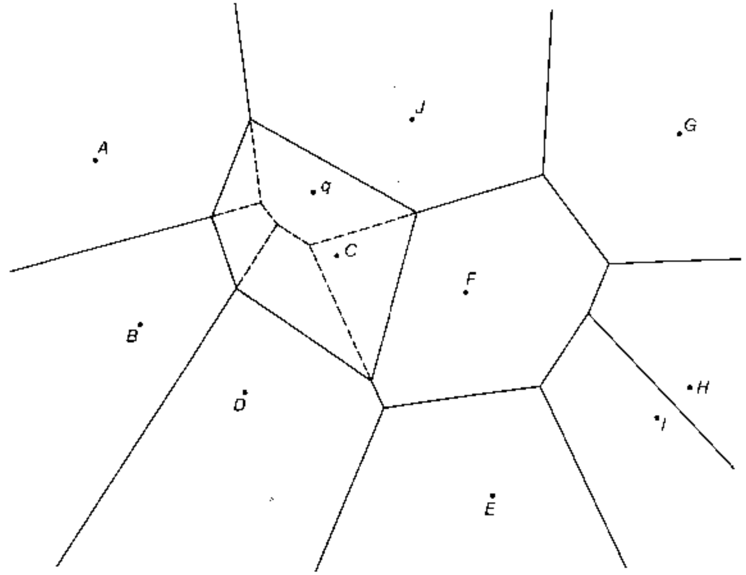


Figure 25. Construction of order-2 Voronoi diagram,

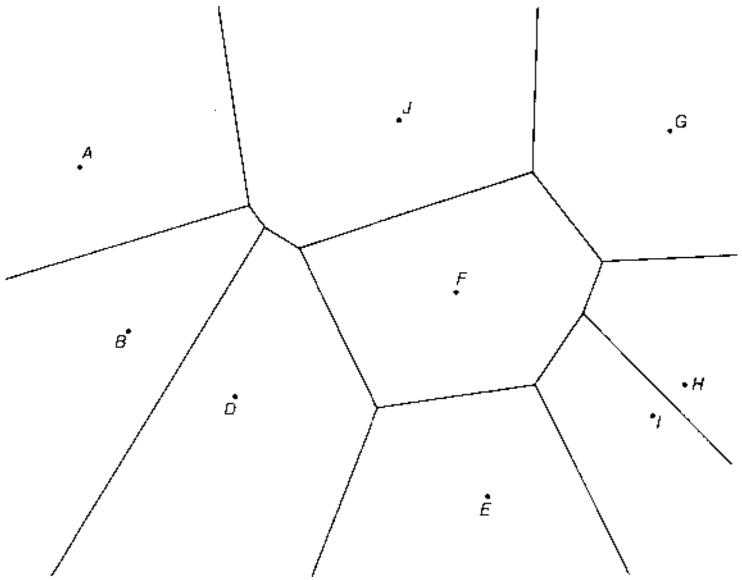


Figure 26 Voronoi diagram of S in Figure 2a with point C removed.

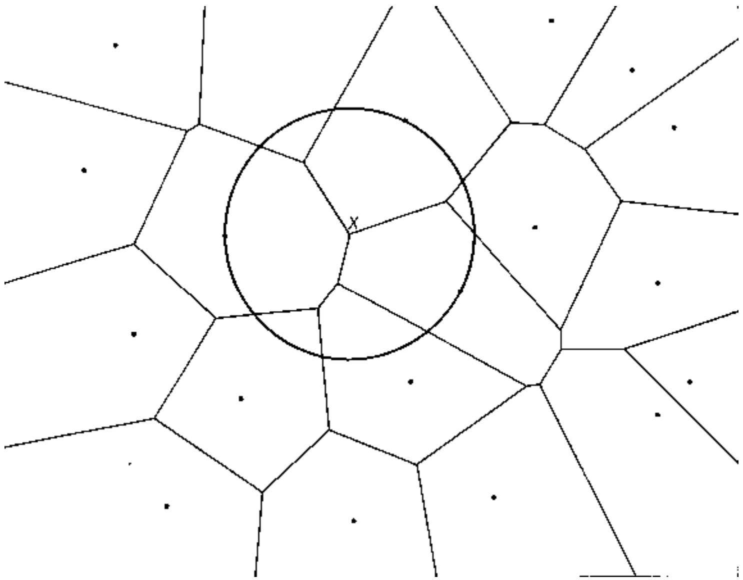


Figure 27. Voronoi diagram showing largest empty circle. X indicates vertex at centre of circle.

single unit emergency depots when more than one unit is required or when the nearest unit is in use (Keeney, 1972).

9.4 Farthest Point Voronoi Diagrams

Again, consider a set of points, S, in the plane. With each point, p_i, in S we associate all locations, x, in the plane which are farther from p_i than any other point p_j ($j \neq i$) in S. This procedure produces a region, F_i, such that

$$F_i = \{x | d(x,i) \geq d(x,j);\ j \in S,\ j \neq i\} \quad (22)$$

Seidel (1982, pp. 25-26) demonstrates that F_i does not exist for all p_i and, in fact, only those points on the convex hull of S (see Section 8.1) have the potential for farthest point regions. The set of all such polygons, F_i, defines the farthest point Voronoi diagram, FV(S), of S. FV(S) for the points in Figure 2a is illustrated in Figure 28. Note that of the points on the convex hull of S (see Figure 13), neither D nor J have further point regions.

Shamos and Hoey (1975) show that FV(S) is related to $V_k(S)$ described in the previous section. This is because the region which consists of all points which are closer to all members of a subset of k points of S, S_k, is also the region of points which are farther from all n-k members of $S-S_k$. Therefore, to solve the order-k farthest point problem we need the solution to the order-(n-k) Voronoi polygon problem. Specifically, FV(S) = V_{n-1} (S).

While FV(S) has obvious utility in the location of obnoxious facilities it is also useful in solving another facilities location problem. This is to find the location, X, where the greatest distance to any point in S is minimized. Such a location is often required in the siting of central emergency facilities. The required location, X, will be the centre of a circle which just encloses the points in S. This circle is either defined by the diameter of S, in whcih case it is determined by two points, or its centre is a vertex of FV(S) in which case it is determined by three points. To find X we consider FV(S) and find the two farthest points in S by computing the distance between all pairs of points in FV(S). The maximum of these distances is the diameter of S. If the circle so defined encloses S we have our solution. If not Shamos and Hoey (1975, p. 161) show that the required centre is a vertex of FV(S). Figure 28 provides the solution for the points shown in Figure 2a.

9.5 Non-Euclidean Distance

Throughout this monograph, whenever we have considered distance we have done so in terms of euclidean (i.e., straight line) distance. However, euclidean distance is only a particular, albeit in geography a very important, instance of a more general way of measuring distance. This more general concept of distance is usually called the Minkowski p-metric or Lp metric. Given two points in the plane, p_i, p_j, with coordinates (x_i, y_i), (x_j, y_j), respectively, the distance between them, dp (i,j), in the Lp metric is given by

$$dp\,(i,j) = (|x_i - x_j|^p + |y_i - y_j|^p)^{1/p} \quad \text{for } p = 1, 2, \ldots \quad (23)$$

Thus, euclidean distance can be considered a special case of the Lp metric when p=2.

Another instance of the Lp metric which may be of interest to geographers occurs when p=1. This yields the so called 'Manhattan' or 'taxicab' distance since it corresponds to a situation in which the points are located on a regular square grid and distance between points is measured along the lines of the grid. It has often been argued that such a measure of distance is more appropriate than the euclidean one in applications in urban areas especially in North America or in rural situations in much of mid-western North America and elsewhere.

Voronoi polygons can be defined using the concept of Manhattan distance rather than euclidean distance. As previously, assume we have a set, S, of n labelled points in the plane and that the distance between any point, p_i, in S and any location, x, in the plane in terms of Manhattan distance is $d_M(x,i)$. With each point, p_i, in S we can create a region, MP_i, consisting of all locations in the plane which are closer to p_i, in terms of Manhattan distance, than to any other point, p_j $(j \neq i)$. Thus,

$$MP_i = \{x \mid d_M(x,i) \leq d_M(x,j);\ j \in S,\ j \neq i\} \qquad (24)$$

The set of all such regions, MP_i, so formed defines the Voronoi diagram of S, MV(S), in Manhattan space. MV(S) for the points shown in Figure 2a is illustrated in Figure 29. Note that the individual polygons so produced in MV(S) are not convex, that their edges consist of only horizontal, vertical and 45 degree line segments and that an individual edge consists of at most three line segments. If for any two points, p_i, p_j $(i \neq j)$, $x_i = x_j$ or $y_i = y_j$, the locus of locations equidistant from p_i, p_j degenerates into a single horizontal line $(x_i = x_j)$ or a single vertical line $(y_i = y_j)$. Also, in Manhattan space when $|x_i - x_j| = |y_i - y_j|$ the locus of locations equidistant from p_i, p_j is not unique. As Figure 30 shows this consists of two regions (shaded)and a line segment. Note that this is the situation for points H and I in Figure 29. In this instance the problem was solved arbitrarily by equally dividing the shaded areas between the two points.

These and other properties of Voronoi diagrams defined in Manhattan space are described by Hwang (1979), Lee (1980) and Lee and Wong (1980). In addition, Hwang (1979) explores the minimal spanning tree (see Section 8.2) of S in Manhattan space. Carter *et al.* (1972) offer a simple example in terms of defining response areas for emergency units in urban areas while Von Hohenbalken and West (1984b) use them in defining supermarket trade areas in Edmonton, Canada.

Although only the Manhattan alternative to euclidean distance has been considered so far, it is possible that Voronoi polygons defined using other distance concepts such as those associated with time or cost measures may well prove of more interest to geographers.

ACKNOWLEDGEMENTS

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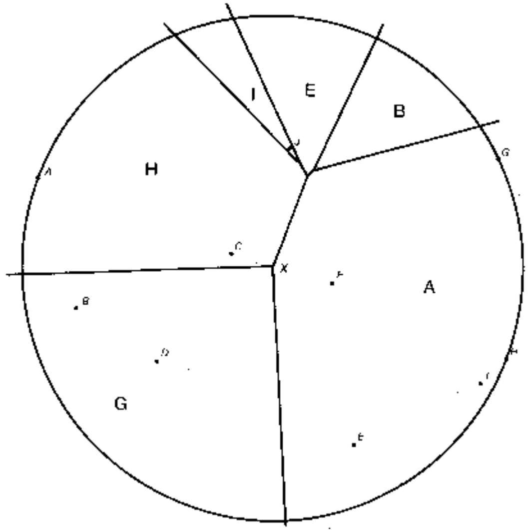

Figure 28. Farthest point Voronoi diagram of S in Figure 2a. Boldface capitals show farthest point regions. X marks vertex at centre of smallest circle enclosing S.

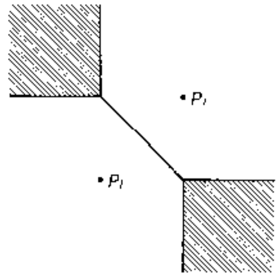

Figure 30. Locus of points equidistant from p_i, p_j.

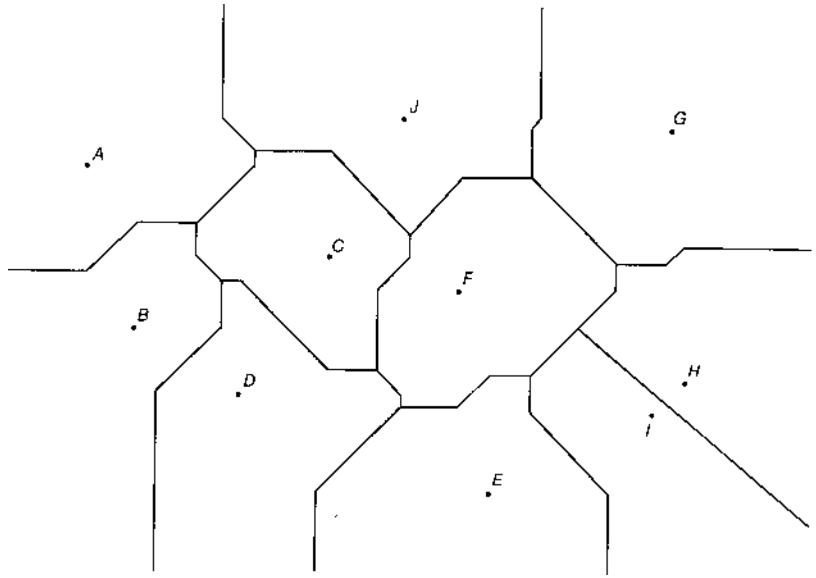

Figure 29. Voronoi diagram of S in Figure 2a. usning Manhattan distance.

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GLOSSARY OF NOTATION

This consists of two parts. The first comprises symbols, with which the reader may be unfamiliar, which define particular mathematical operations or conditions. In the second part, specialized notation which is used throughout the monograph is defined.

Mathematical operators and conditions

$\{\ \}$	the set of, thus $\{a,b,c\}$ is the set with elements a,b,c.
$\in$	is an element of, thus $x \in S$ states that x is a member of the set S.
$\forall$	for all, thus $\forall x$ stands for all x.
$\vert$	such that, thus $\{x \vert x>7\}$ is the set of all x's such that $x>7$.
$\neq$	not equal to, thus $a \neq b$ states that a is not equal to b.
Σ	summation symbol, thus $\sum_i x_i$, is the summation of all values of x_i, and $\sum_{i=1}^{n} x_i$ is the summation of all values of x_i between $i=1$ and $i=n$, inclusive.
$F(x)$	The cumulative probability distribution of a (random) variable, x.
$f(x)$	the probability density function of a continuous random variable, x.
$\int_a^b f(x)dx$	the integral of f from a to b.
e	the mathematical constant, 2.718282
!	factorial, $n!=(n)(n-1)(n-2)\ldots(3)(2)(1)$.

Specialized symbols

S	$=\{p_1, p_2, \ldots p_n\}$, a set of labelled points in the plane.
p_i	the ith point in S.
w_i	the weight $(w_i>0)$ attached to p_i.
P_i	the polygon associated with p_i.
x	an arbitrary location in the plane.
$d(x,i)$	the euclidean distance between x and p_i.
$d_w(x,i)$	the weighted distance between x and p_i.
$V(S)$	the Voronoi diagram of S.
$WV(S)$	the weighted Voronoi diagram of S.

APPENDIX A

Review of the History of the Development of Voronoi Polygons in Different Disciplines

Such polygons were originally described by the mathematician Dirichlet although his treatment (Dirichlet, 1850, especially pp. 216-220) is quite indirect. Nevertheless this led to some people christening them Dirichlet domains (or regions). Later a more straightforward treatment was presented by another mathematician (Voronoi, 1908) and as a result they are most frequently referred to as Voronoi polygons. Shortly after this Thiessen (1911) developed them as an aid in computing more accurate estimates of regional rainfall averages and the name Thiessen polygon became the preferred one in geography. Later still the physicists Wigner and Seitz (1933) defined them in three dimensions (hence their common name in physics of Wigner-Seitz regions) only to have two other physicists rediscover them in essentially the same context and to label them as the domains of the atom (Frank and Kasper, 1958). Even as recently as 1966 the ecologist Mead discovered them again, this time coining the term 'plant polygons'. It is quite possible that other more obscure discoveries have been made at other times and places.

APPENDIX B

Examples of the Use of Voronoi Polygons in Disciplines Other than Geography

As noted in the Introduction, Voronoi polygons and their extensions have been used in a diverse range of disciplines. Here we provide brief descriptions of some applications in disciplines other than geography to give the reader some indication of the scope of such applications. The applications are grouped under the same headings used for the sections in the body of the monograph.

Random Voronoi Polygons. These have been used as normative models in studies of territories formed by gregarious animals (Hamilton, 1971a, 1971b) Royal tern nests (Buckley and Buckley, 1977) and male mouth-breeder fish (Hasegawa and Tanemura, 1967; Tanemura and Hasegawa, 1980) and of the dispersion of particles in 0.2 wt%C steel (Wray *et al.,* 1983).

Data Manipulation. Several researchers have observed that Voronoi polygons provide good approximations of cellular patterns, including monolayer cells and epithelial cells in tissue (Honda, 1978; Honda at al., 1979; Saito, 1982) and some metals (Wray *at al.,* 1983).

Model of Spatial Processes. Voronoi polygons have been used as assignment models in archaeology and anthropology. Here they have been used as surrogates for the territories of Mayan ceremonial centres (Hammond, 1972, 1974), Romano-British walled centres (Hodder, 1972), hillforts (Cunliffe, 1971), neolithic long barrows (Renfrew, 1973a) and Maltese temples (Renfrew, 1973b). Similarly, Callen and Stephen (1975) used Voronoi polygons to simulate Siwai village boundaries on Bougainville Island in the Solomons.

Gabriel Graph. In biology the definition of connectivity provided by the Gabriel graph has been used to study gene flow. Matula and Sokal (1980) provide a review of such work.

Weighted Voronoi Polygons. Archaeologists have used structures formed using the weighting scheme in expression (17) to define territories for various sites. Their work in this regard is reviewed by Hodder and Orton (1976, pp. 187-195). The most extensive of such studies is that of Hogg (1971) who used weighted Voronoi polygons as surrogates for territories of neolithic hillforts south of the River Thames in England.

Weighted Voronoi polygons formed using the weighting scheme in expression (19) have been used in modeling the crystal structures of organic compounds (Fischer and Koch, 1979) and metallic glass (Gellatly and Finney, 1982).

Generalized Voronoi Polygons. Kirkpatrick (1979) has examined Voronoi polygons associated with closed polygonal figures, while Drysdale and Lee (1978) and Lee and Drysdale (1981) considered Voronoi polygons associated with disjoint (unconnected) line segments and with circles. In the case of unconnected line segments the edges of the generalized Voronoi polygons are composites of straight-lines and pieces of parabolas.

APPENDIX C

Availability of Computer Programs to Generate Voronoi Polygons and Delaunay Triangles

As noted in the Introduction a number of algorithms exist for constructing Voronoi polygons and related structures in two and three dimensions. These algorithms and their related programs can often be obtained from their originators. The most extensive of these programs is TILE4 (Green and Sibson, 1978) which is part of a set of routines for spatial analysis, available (for a small fee) from Professor Sibson (School of Mathematics, University of Bath, Bath BA2 7AY, U.K.).

The author of this monograph also has available a program which includes an algorithm developed by Nigel Horspath and which constructs both the Voronoi diagram and Delaunay triangulation of a set of points. This program also identifies and measures the minimum angle of each triangle in the Delaunay triangulation as well as identifying those triangles where one or more vertices form one of the vertices of the outer boundary of the triangulation (i.e. the convex hull of the points). Such information is useful in accounting for edge effects. A listing of this program in either FORTRAN 77 or Macintosh Microsoft BASIC can be obtained free of charge from the author.

If preferred either a tape or a disc can be supplied at cost. In both versions the program contains a full description of the program and its operation. Test data will also be supplied.

A program, EQUAL, for constructing the weighted Voronoi diagram using the weighting scheme in expression (17) has been written by Robert Wittick. A FORTRAN version of this program is available at cost through the Geography Program Exchange (Department of Geography, Michigan State University, East Lansing, Michigan 48824, USA). A Macintosh Microsoft BASIC version can be obtained (in either listing or disk form) from the author of this monograph.

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